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Geometric study of the connection between the Lagrangian and Hamiltonian constraints

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Abstract. *The second order differential equation character of the solutions of the dynamical equation $i(\Gamma)\omega_L = \mathrm{d}E_L$ for a singular Lagrangian L, as well as the conditions for the existence of such a solution, are studied. We also introduce a couple of maps $R(L)_v : T_{FL(v)}(T^*Q) \to T_v(TQ)$ and $T(L)_v : T_{FL(v)}(T^*Q) \to T_{FL(v)}(T^*Q)$, with $v \in TQ$, which are shown to be very useful for establishing the connection between the constraints arising in the Lagrangian and Hamiltonian formulations.*

1. INTRODUCTION

The geometric approach to different problems in some branches of Theoretical Physics, for instance Classical Mechanics and Classical Field Theory, seems to be very suitable and fruitful and it has given rise to a considerable progress in the understanding of some aspects of Lagrangian and Hamiltonian systems. In particular, the Symplectic Geometry has been shown to be the appropriate framework for the description of autonomous regular mechanical systems providing a way for dealing on the same footing with Lagrangian and Hamiltonian systems (see e.g. [1, 2]). There is however a lot of interesting physical systems that are described

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by singular Lagrangians and the study of such systems, started by Dirac ([3, 4]), is receiving much attention on the last years, no doubt because these systems exhibit gauge invariance, a crucial property in the modern physical theories. From the geometric point of view, it is clear the necessity of a more general framework to take account of such singular systems and the geometric approach developed by several authors (see e.g. [5, 10] and for a particular account, see [11]) seems to be worth a deeper analysis on some specific points as the relation between the constraint functions in the Lagrangian and Hamiltonian formulations respectively, as well as the second order differential equation (hereafter shortened as S.O.D.E.) problem.

More accurately, in the traditional approach, when the Lagrangian is singular the Euler-Lagrange equations cannot be written in normal form because the matrix of the coefficients of the accelerations is singular and then either the solutions of such equations are not uniquely determined by the initial conditions, or even worse, there is no solution at all for some particular choices of the initial conditions. In the geometric approach, some of these aspects are clarified, but more to the point is that the Euler-Lagrange equations do not correspond to some of the solutions of the dynamical equation ([7, 12]). This is a very interesting feature of the singular Lagrangian systems which was pointed out by Gotay and Nester and that unfortunately seems to have been unnoticed for the most part of physicists. In such cases the solution of the dynamical equation can only be chosen to be the restriction of a S.O.D.E. in some points of the velocity phase space and thus some additional constraints in the Lagrangian formulation arise in this way. In other words, besides the constraint functions determining the points where there exist solutions of the dynamical equation, wich will be called dynamical constraints, there are other constraint functions determining the subset where a solution restriction of a S.O.D.E. can be found. These last constraints will be called S.O.D.E. conditions.

On the other hand, a Hamiltonian formulation is also possible for systems described by singular Lagrangians, and this was actually the form in which Dirac introduced his theory of constraints. In this situation the image of the velocity phase space TQ under the Legendre transformation $FL : TQ \to T^*Q$ does not cover all the phase space T^*Q, but only a submanifold M_1, $j : M_1 \to T^*Q$, called the primary constraint submanifold. The pull back on M_1 of the canonical symplectic form Ω defined in the phase space, assumed to be of a constant rank, gives a presymplectic form and as the energy E_l is a FL-projectable function, i.e., there exists a function $H \in C^\infty(M_1)$ such that $H \circ FL = E_L$, a presymplectic system $(M_1, j^*\Omega, H)$ can then be defined. The general theory of presymplectic systems ([5 - 7, 13]), geometric version of Dirac's theory, can therefore be applied either to (TQ, ω_L, E_L) or to $(M_1, j^*\Omega, H)$ giving rise to constraint functions. The relation between dynamical Lagrangian and Hamiltonian constraint functions is well known [6]: essentially the dynamical Lagrangian constraints arise as the FL-pull-back of secondary Hamiltonian constraints.

However, the usual study of Lagrangian constraints takes as a starting point the Euler-Lagrange equations and some of the constraints so obtained have no counterpart in the geometric version when the presymplectic system (TQ, ω_L, E_L) is considered. Actually, the additional constraints that appear correspond to the S.O.D.E. conditions described in a former paragraph. Moreover, the S.O.D.E. conditions can be shown to be non-FL-projectable [14] while the dynamical constraints associated with such a system are just the projectable Lagrangian constraints.

In very recent papers ([15, 16]), the relation between all the constraint functions arising in the Lagrangian and Hamiltonian formulations is studied, but their approach was a local, coordinate dependent, one. This paper aims to complete the preceding papers by carrying out a geometrical analysis with no use of coordinate expressions which could mask the coordinate-dependent nature of some local expressions. Moreover, the intrinsic character of the expressions can in no way be considered as an «academic subject», but as a step more in the process of geometrization of the physics; as remarked by Lichnerowicz [17] «if we understand truly classical analytical dynamics we have a chance to understand more easily quantum dynamics and to obtain new invariant tools». Actually, the fact that the expressions are intrinsic permits a generalization to infinite-dimensional systems.

We will show that the non-dynamical Lagrangian constraints arise in a natural way when the S.O.D.E. character of the solutions is taken into account. The most important contribution of the paper is that in order to relate the set of the Hamiltonian constraint functions with that of all the Lagrangian ones we are going to show that it can not be used the FL-pull-back which only would give the FL-projectable constraints, but we develop an alternative method based on the introduction of a map R pulling the vector fields in M_1 back to TQ and establishing in this way the appropriate relation.

The paper is organized as follows: In Section 2 we introduce the notation and the basic definitions to be used later and exhibit the S.O.D.E. problem for singular Lagrangians. Section 3 is devoted to study the relations between Ker ω_L and its vertical part Ker FL_*, as well as whether the image of Ker ω_L under the vertical endomorphism S covers Ker FL_*. The distinction between dynamical constraints and S.O.D.E. conditions in the Lagrangian formulation is given in Section 4. In order to connect this theory with the Hamiltonian one, we introduce in Section 5, in a pure geometrical way, two L-dependent maps which allow to give a geometrical sense to some of the results obtained in previous papers ([15, 16, 18 and

19]) using coordinate expressions. Finally the theory is get ready for explicit calculations and applications in the last Section where the results of the paper are also displayed by means of some physical examples.

2. NOTATION AND BASIC DEFINITIONS

Time-independent Lagrangian svstems are described from the geometric point of view using the geometry of the tangent bundle, TQ, of a differentiable manifold Q, the configuration space of the system. We will use through this paper the results of the paper by Crampin [20] concerning the geometry of the tangent bundle and particularly the properties of the fundamental (1, 1) canonical tensor S, called the vertical endomorphism, which defines the natural almost tangent structure of TQ. Given a function $L \in C^\infty(TQ)$, we define an exact two-form $\omega_L = -\,\mathrm{d}(\mathrm{d}L \circ S)$ and a function $E_L = \Delta(L) - L$, called the energy function, where $\Delta \in \mathscr{X}(TQ)$ denotes the Liouville vector field generating dilations along the fibres of TQ. If ω_L is of a constant rank, L is called the Lagrange function, and we can consider the presymplectic system given by the triplet $(TQ, \omega_L, \mathrm{d}E_L)$ with the associated equation

$$i(\Gamma)\omega_L = \mathrm{d}E_L \tag{2.1}$$

whose solutions determine the possible dynamics.

The particular case of L being such that ω_L is of maximal rank, i.e. the bundle map, $\hat{\omega}_L : T(TQ) \to T^*(TQ)$, defined by contraction, is invertible, is that of regular Lagrangians and the Legendre transformation [1], $FL : TQ \to T^*Q$, is then a local diffeomorphism. Moreover, in this last case, it is easy to see that the dynamical equation (2.1) has only a solution which is a second order differential equation field, i.e., $S(\Gamma_L) = \Delta$, and moreover the curves projection on Q of the integral curves of Γ_L will satisfy the well-known Euler-Lagrange equations. It was remarked [7] that this fact does not hold for a singular Lagrangian. In fact, if L is singular the two-form ω_L is not symplectic and the dynamical equation (2.1) may not have solution in some points and the solution is not unique in the other points, and on the projection on Q of the integral curves of such a solution, the Euler-Lagrange equations do not hold. Similarly, we can consider the presymplectic system defined on the image of TQ under FL, $j : FL(TQ) \equiv M_1 \to T^*Q$, $(M_1, j^*\Omega, \mathrm{d}H)$, where H is the FL-projection of the energy function and Ω is the canonical symplectic form on T^*Q, with the associated equation

$$i(X)j^*\Omega = \mathrm{d}H \tag{2.2}$$

Gotay *et al.* [5 - 7] have developed a geometric algorithm for the determination of a maximal submanifold in both cases, called the final constraint submanifold

$C \subset TQ$ (respectively $P \subset T^*Q$), in which the dynamical equation (2.1) (respectively (2.2)) has a consistent solution.

In a recent paper [12] it was shown that for a particular type of Lagrangians, called Type II Lagrangians in [21], it is possible to find a second order differential equation whose restriction on the final constraint submanifold satisfies the dynamical equation. The proof is purely algebraic and it does not assure wether the S.O.D.E. solution is tangent to C or not. In the general case we cannot guarantee the existence of a solution restriction on C of a S.O.D.E. and then, some additional S.O.D.E. conditions selecting the subset of C in which such a solution can be found must be considered.

The proofs of these facts are based on the following properties connecting S and the presymplectic form ω_L [20]:

$$\text{(2.3a)} \qquad \text{i)} \quad i(S(U))\omega_L = - i(U)\omega_L \circ S, \quad \forall U \in \mathfrak{X}(TQ),$$

$$\text{(2.3b)} \qquad \text{ii)} \quad i(\Delta)\omega_L = - \mathrm{d}E_L \circ S.$$

We will denote Ker ω_L the subbundle of $T(TQ)$

$$\operatorname{Ker} \omega_L = \{U \in T(TQ) \mid i(U)[\omega_L(\pi_{TQ}(U))] = 0\},$$

but, with some abuse of notation, we also denote Ker ω_L the set of the smooth sections of $\pi_{TQ} : T(TQ) \to TQ$ taking values in the subbundle Ker ω_L. Similarly, we also denote $\hat{\omega}_L$ the map from $\mathfrak{X}(TQ)$ into $\Lambda^1(TQ)$ obtained by making act $\hat{\omega}_L$ on sections of $\pi_{TQ} : T(TQ) \to TQ$. With this notation, the relation (2.3a) may be written $\omega_L \circ S = - S^* \circ \hat{\omega}_L$, which was introduced for regular Lagrangians by Vershik and Faddeev [22].

The above properties (2.3) can be used to prove that $S(\operatorname{Ker} \omega_L) \subset \operatorname{Ker} \omega_L \cap \cap V(TQ) \equiv V(\operatorname{Ker} \omega_L)$, where $V(TQ) \equiv \operatorname{Ker} \pi_{Q*}$ is the subbundle of vertical vectors in $T(TQ)$. The set of sections of the subbundle $V(TQ)$ will be denoted $\mathfrak{X}^V(TQ)$ and its elements are called vertical fields. For any vector field Γ solution of the dynamical equation (2.1), $S(\Gamma) - \Delta \in V(\operatorname{Ker} \omega_L)$. In particular, as pointed before, when ω_L is regular Γ_L is a S.O.D.E.. On the other hand, we remark that in the general case, another vector field Γ' will also be solution of (2.1) if and only if the difference $\Gamma - \Gamma'$ lies in Ker ω_L. Similarly, if Γ is a S.O.D.E., Γ' is a S.O.D.E. if and only if the difference $\Gamma' - \Gamma$ is a vertical field. Thus, the idea is to start from a particular solution Γ of (2.1) and to modify it by adding an element in Ker ω_L in order to obtain, if possible, a S.O.D.E. solution of (2.1) too. It will be possible if the difference $S(\Gamma) - \Delta$ is the image under S of an element in Ker ω_L. Then, we will study the image of Ker ω_L under S and we will analyse the cases when the map $S' \equiv S|_{\operatorname{Ker} \omega_L}$ is onto $V(\operatorname{Ker} \omega_L)$.

3. PROPERTIES OF Ker ω_L AND Ker FL_*

In order that this paper be selfcontained we begin this section by reviewing the well-known property $V(\text{Ker } \omega_L) = \text{Ker } FL_*$, giving an intrinsic, coordinate-independent, proof. Other properties related to the map $S' : \text{Ker } \omega_L \to V(\text{Ker } \omega_L)$ will also be given.

PROPOSITION 1. *Let v be a vector $v \in TQ$. The tangent space in $FL(v)$, $T_{FL(v)}(T^*Q)$, is a sum of the two vector subspaces $V_{FL(v)}(T^*Q)$ and $FL_{*_v}[T_v(TQ)]$, namely,*

$$T_{FL(v)}(T^*Q) = V_{FL(v)}(T^*Q) + FL_{*_v} T_v(TQ). \tag{3.1}$$

Proof. We first recall that FL is a base preserving bundle map, i.e., $\tau_0 FL = \pi$, where $\tau : T^*Q \to Q$ and $\pi : TQ \to Q$ are the corresponding projections of the cotangent and the tangent bundles respectively. Therefore, for any $Z \in T_{FL(v)}(T^*Q)$, as π is a submersion, there will exist a vector $Y \in T_v(TQ)$ such that $\tau_{*_{FL(v)}} Z = \pi_{*_v} Y$. It is an easy task to check that $Z - FL_{*_v} Y = W$ is a vertical vector $W \in V_{FL(v)}(T^*Q)$, so that any $Z \in T_{FL(v)}(T^*Q)$ can be decomposed as a sum $Z = FL_{*_v} Y + W$ and the statement of the proposition follows. ∎

PROPOSITION 2. *Given any $v \in TQ$, the subspaces $V_{FL(v)}(T^*Q)$ and $[FL_{*_v} T_v(TQ)]^\perp$ are disjoint,*

$$V_{FL(v)}(T^*Q) \cap [FL_{*_v} T_v(TQ)]^\perp = 0. \tag{3.2}$$

Proof. It is just a consequence of Proposition 1, when taking into account that $[V_{FL(v)}(T^*Q)]^\perp = V_{FL(v)}(T^*Q)$, where the symbol $\perp$ makes reference to the orthogonal with respect to the canonical symplectic structure Ω in T^*Q, i.e., for every subbundle H of $T(T^*Q)$, $H^\perp$ will denote $H^\perp = \{Z \in T(T^*Q) \mid \Omega(Z, X) = 0 \text{ for every } X \in H\}$. ∎

The symbol $\perp$ will be used either with respect to the symplectic structure Ω in T^*Q or with respect to the presymplectic structure ω_L in TQ.

PROPOSITION 3. *The vertical part of* Ker ω_L *coincides with* Ker FL_*

$$V(\text{Ker } \omega_L) = \text{Ker } FL_* \tag{3.3}$$

Proof. First of all, since ω_L coincides with the pull-back of Ω under FL, we see that $\text{Ker } FL_* \subset \text{Ker } \omega_L$ and $FL_*(\text{Ker } \omega_L) \subset [FL_*(TTQ)]^\perp$. Moreover, as FL is fibre-preserving, $FL_*(V(TQ)) \subset V(T^*Q)$. Then,

$$FL_*(V(\text{Ker } \omega_L)) = FL_*(V(TQ) \cap \text{Ker } \omega_L) \subset [FL_*(TTQ)]^{\perp} \cap V(T^*Q) = 0,$$

so that the relation $V(\text{Ker } \omega_L) \subset \text{Ker } FL_*$ follows.

Conversely, from $\tau \circ FL = \pi$ we can see that Ker FL_* is made up by vertical vectors, so that Ker $FL_* \subset$ Ker $\omega_L \cap V(TQ) = V(\text{Ker } \omega_L)$ and Proposition 3 is proved. ∎

DEFINITION 1. Let M be the subset of $T(TQ)$ determined by the inverse image of $V(\text{Ker } \omega_L)$ under S,

$$M = \{U \in T(TQ) \mid S(U) \in V(\text{Ker } \omega_L)\}. \tag{3.4}$$ ∎

It is now an easy task to check that $M = [V(TQ)]^{\perp}$ because of relation (2.3a). Moreover, M is coisotropic, i.e., $M^{\perp} \subset M$, and every one form image of a vertical vector field under $\hat{\omega}_L$ vanishes on M.

PROPOSITION 4. *For every X in $M^{\perp}$ there exists a vertical vector V such that $X - V \in$ Ker ω_L. Such vector V is only defined up to an element of $V(\text{Ker } \omega_L)$.*

Proof. Given an arbitrary vector X, the equation $i_V \omega_L = i_X \omega_L$ can be solved for V vertical if and only if $\forall U \in [V(TQ)]^{\perp}$, $\omega_L(X, U) = 0$. In particular, this is true if X lies in $M^{\perp}$ because $[V(TQ)]^{\perp} \equiv M$. For any couple V_1, V_2 of such vertical fields the difference is vertical and lies in Ker ω_L. ∎

This gives an identification of $M^{\perp}$ with Ker ω_L up to elements of $V(\text{Ker } \omega_L)$ and an immediate consequence of Proposition 4 is that if $M^{\perp}/S$ is the factor bundle defined by the equivalence relation associated to $S|_{M^{\perp}}$, $M^{\perp}/S \equiv \text{Ker } \omega_L/S$, and next commutative diagram follows:

$$\begin{array}{ccc} M^{\perp}/S \equiv \text{Ker } \omega_L/S & \leftrightarrow & S(\text{Ker } \omega_L) \\ \downarrow & & \downarrow \\ M/S & \leftrightarrow & V(\text{Ker } \omega_L) \\ \downarrow & & \downarrow \\ TTQ/S & \leftrightarrow & V(TQ) \end{array} \tag{3.5}$$

the vertical arrows meaning the natural injections and the horizontal identifications being given by S.

COROLLARY 1. *A vertical vector X is in $S(\text{Ker } \omega_L)$ if and only if for every Z such that $S(Z) = X$ and every U in M, $\omega_L(Z, U) = 0$.* ∎

It is worth recalling that it has recently beeen shown [12] that the map $S' \equiv$ $\equiv S|_{\text{Ker}\,\omega_L}$, S' : Ker $\omega_L \to V(\text{Ker } \omega_L)$ is onto if and only if

$$(3.6) \qquad \dim \operatorname{Ker} \omega_L = 2 \dim V(\operatorname{Ker} \omega_L),$$

and consequently, for such Lagrangians, if X is a solution of (2.1) on the final constraint submanifold C, since $S(\Gamma) - \Delta \in V(\operatorname{Ker} \omega_L)$, there will exist $U \in \operatorname{Ker} \omega_L$ such that $S(X) - \Delta = S(U)$. Then, $\Gamma = X - U$ is a S.O.D.E. satisfying the dynamical equation on C. However we are not able to guarantee that Γ can be chosen as being tangent to the final constraint submanifold C unless the dynamics is global (all the constraints are trivial and $C = TQ$). Another characterization of such Lagrangians, called type II Lagrangians, is given by next Corollary wich also follows from the former diagram.

COROLLARY 2. $S(\operatorname{Ker} \omega_L) = V(\operatorname{Ker} \omega_L)$ *if and only if* $M = M^\perp$. ∎

4. THE PRIMARY LAGRANGIAN CONSTRAINTS AND THE EQUATIONS OF MOTION

As indicated above, if the Lagrangian L is singular the two-form ω_L is presymplectic and as a first step we have to determine the points of TQ in which the dynamical equation has a solution. It is well known that the compatibility condition is [7]

$$(4.1) \qquad i(Z)\, \mathrm{d}E_L = 0, \quad \forall Z \in \operatorname{Ker} \omega_L .$$

We remark that for any S.O.D.E., Γ_0, the one-form $i(\Gamma_0)\omega_L - \mathrm{d}E_L$ is horizontal, i.e. semibasic [20]. In fact, this property is equivalent to

$$(4.2) \qquad (i(\Gamma_0)\omega_L - \mathrm{d}E_L) \circ S = 0,$$

and this follows from (2.3) after a little of algebra. The converse property is only true for regular Lagrangians but not for a singular one. Property (4.2) can be used to show that condition (4.1) is automatically satisfied for any $Z \in V(\operatorname{Ker} \omega_L)$ as indicated by Cantrijn [23], just considering an arbitrary $X \in M$ such that $S(X) = Z \in V(\operatorname{Ker} \omega_L)$, and contracting it with (4.2).

THEOREM 1. *Given a Lagrangian function* $L \in C^\infty(TQ)$, *the three next assertions are equivalent:*

i) The dynamical equation (2.1) admits a global S.O.D.E. solution.

ii) There exists a global solution of (2.1) and for every such solution X *and each S.O.D.E.* Γ_0, *the difference* $X - \Gamma_0$ *is such that its image under* $\hat{\omega}_L$ *vanishes on the elements of* M.

iii) For every S.O.D.E. Γ_0, *the horizontal one-form* $i(\Gamma_0)\omega_L - \mathrm{d}E_L$ *is in the image under* $\hat{\omega}_L$ *of the vertical vectors, or in an equivalent formulation, the one-form vanishes on the vectors in* M.

Proof. Let Γ_0 denote an arbitrary S.O.D.E. and let us suppose that i) is true, i.e. there exists a global S.O.D.E. solution Γ. Then for every global solution X and every S.O.D.E. Γ_0

$$\hat{\omega}_L(X-\Gamma_0) = \mathrm{d}E_L - \hat{\omega}_L(\Gamma_0) = \hat{\omega}_L(\Gamma-\Gamma_0)$$

and $\Gamma - \Gamma_0$ being vertical, property ii) follows.

Beginning with property ii) as an hypothesis and considering a global solution X of (2.1), the expression $\hat{\omega}_L(X-\Gamma_0) = \mathrm{d}E_L - \hat{\omega}_L(\Gamma_0)$ shows that $i(\Gamma_0)\omega_L - - \mathrm{d}E_L$ is in the image of the vertical vector fields under $\hat{\omega}_L$.

Finally, if $i(\Gamma_0)\omega_L - \mathrm{d}E_L$ vanishes when applied to M, it means that the equation $i(\Gamma_0)\omega_L - \mathrm{d}E_L = i(V)\omega_L$ can be solved for V vertical and then $\Gamma_0 - V$ is a global S.O.D.E. solution. ∎

Using a procedure similar to the one used in the derivation of the Theorem 1, we can conclude that the set of points of TQ in which a solution of (2.1) that is the restriction of a S.O.D.E. exists, is made up by the points in which the one-form $i(\Gamma_0)\omega_L - \mathrm{d}E_L$ takes values in $\hat{\omega}_L(V(TQ))$ for one S.O.D.E. Γ_0 (and therefore for any S.O.D.E. too), or in an equivalent formulation, it vanishes on the vectors in M. Alternatively, it can also be defined as the set of points of the primary dynamical Lagrangian constraint submanifold in which the image under $\hat{\omega}_L$ of the difference $X - \Gamma_0$ vanishes on the corresponding elements of M, X and Γ_0 being an arbitrary solution of (2.1) in that points and a S.O.D.E., respectively. In order to find a method for obtaining the subset in which a solution of the dynamics restriction of a S.O.D.E. exists, we remark that if U lies in M, the condition obtained for U is the same as for $U + S(Y)$, so that we must choose just one element in every fibre of $M \to M/S$ and the corresponding condition reads

$$(\hat{\omega}_L(\Gamma_0) - \mathrm{d}E_L)(U) = 0, \tag{4.3}$$

for an arbitrary S.O.D.E.. This condition is no matter of the choice of Γ_0 and in the particular case in which U can be chosen in Ker ω_L this is not an additional condition, but it reduces to a condition for the existence of a solution of (2.1). If the map $S' = S|_{\mathrm{Ker}\,\omega_L}$ is onto, we can also see that the S.O.D.E. condition is automatically satisfied. In fact, if X and Γ_0 are as in the former paragraph, then the values of the difference $X - \Gamma_0$ have to lie in M because

$$i(S(X-\Gamma_0))\omega_L = i(S(X))\omega_L - i(\Delta)\omega_L = i(X)\omega_L \circ S - i(\Delta)\omega_L = 0,$$

the last relation coming from (2.3b) and so if S' is onto, the result of the Corollary 2 shows that $\hat{\omega}_L(X-\Gamma_0)$ vanishes on M.

In summary, the constraint functions defining the subset of TQ where the equation (2.1) has a solution that is a restriction of a S.O.D.E. are in a one-to-one correspondence with the sections of $V(\mathrm{Ker}\ \omega_L) = M/S$, by means of (4.3), the dynamical constraint functions being associated with sections of $S(\mathrm{Ker}\ \omega_L)$ while the additional S.O.D.E. conditions are associated with the other ones.

5. THE CORRESPONDENCE WITH THE HAMILTONIAN FORMULATION

DEFINITION 2. Let $\eta : E \to Q$ be a vector bundle on Q and $F : TQ \to E$ a bundle map preserving the base Q. We will denote $R(F)$ the map $R(F) : \mathscr{X}(E) \to \mathscr{X}^V(TQ)$, given by

$$[R(F)X]_u = \xi^u\,[\eta_{*F(u)} X_{F(u)}], \qquad \forall X \in \mathscr{X}(E) \tag{5.1}$$

where $\xi^u : T_{\pi(u)}Q \to T_u(TQ)$ is the vertical lift

$$\xi^u(w)f = \mathrm{d}(f(u + tw))/\mathrm{d}t\,|_{t=0}, \quad \forall f \in C^\infty(TQ). \tag{5.2}$$ ■

This vector field is differentiable. In fact, its nonvanishing components are the pull-back by F of the first components of X.

A particular case of such definition is when $E = TQ$ and F is the identity map in which $R(id_{TQ})$ reduces to the well-known vertical endomorphism S. We will deal with the case $E = T^*Q$ and F the Legendre transformation $FL : TQ \to T^*Q$ associated to a given Lagrangian function L. The notation $R(L)$ will be used instead of $R(FL)$. We also remark that a map $R(L)$ of $T(T^*Q)$ in $T(TQ)$ inducing the corresponding map $\mathscr{X}(T^*Q) \to \mathscr{X}(TQ)$ does not exist. However, for any $v \in TQ$ we can define a map

$$R(L)_v : T_{FL(v)}(T^*Q) \to T_v(TQ),$$

given by

$$R(L)_v(U) = \xi^v[\tau^*_{FL(v)} U], \qquad \forall\, U \in T_{FL(v)}(T^*Q). \tag{5.3}$$

We next list a set of interesting properties of $R(L)$.

PROPOSITION 5. *i) The image under $R(L)_v$ is in the subspace of vertical vectors $V_v(TQ)$ and correspondingly the vector fields $R(L)X$ are vertical for any $X \in \mathscr{X}(T^*Q)$.*

ii) The following relation holds: $R(L)_v \circ FL_{*v} = S_v$.

Proof. i) is a consequence of the definition (5.3) and ii) follows from the relation $\tau \circ FL = \pi_Q$, which implies $\tau_{*FL(v)} \circ FL_{*v} = \pi_{Q*v}$. ■

A new map $T(L)_v : T_{FL(v)}(T^*Q) \to T_{FL(v)}(T^*Q)$ is also going to play an important role in the description of the relation between the Lagrangian and the Hamilotnian formulations.

DEFINITION 3. Let $T(L)_v$ be the map $T(L)_v : T_{FL(v)}(T^*Q) \to T_{FL(v)}(T^*Q)$, given by

$$T(L)_v X = FL*_v R(L)_v(X), \quad \forall X \in T_{FL(v)}(T^*Q). \tag{5.4}$$ ■

It is obvious that the image of $T(L)_v$ is in $V_{FL(v)}(T^*Q)$. On the other hand, there is an interesting relation between $T(L)_v$ and the vertical endomorphism: $FL*_v$ intertwines $T(L)_v$ and S_v.

PROPOSITION 6. *$T(L)_v$ and S_v are connected by the following relation:*

$$T(L)_v \circ FL*_v = FL*_v \circ S_v. \tag{5.5}$$

Proof. Both sides of the relation coincide as a consequence of ii) in Proposition 5: $T(L)_v \circ FL*_v = FL*_v \circ R(L)_v \circ FL*_v = FL*_v \circ S_v$. ■

In the particular case of a hyper-regular Lagrangian in which FL would be a global diffeomorphism, this shows that $T(L)$ is the (1.1) tensor field in T^*Q corresponding to the vertical endomorphism S in TQ. There is a relation playing a role similar to (2.3a) which will be given in the following proposition:

PROPOSITION 7. *If Ω denotes the canonical symplectic form in T^*Q, then*

$$[i(T(L)_v X)\Omega](FL(v)) = -[i(X)\Omega] \circ T(L)_v, \quad \forall X \in T_{FL(v)}(T^*Q) \tag{5.6}$$

Proof. It is based on the relation (3.1) according to which $T_{FL(v)}[T^*Q]$ can be written as a sum

$$T_{FL(v)}[T^*Q] = V_{FL(v)}(T^*Q) + FL*[T_v(TQ)]$$

and that the restriction of Ω on the subbundle of vertical vectors vanishes. It is enough to prove (5.6) when applied to a vector either in $V_{FL(v)}(T^*Q)$ or else in $FL*_v T(TQ)$, because of the linearity of the expression (5.6)

i) If $Y \in V_{FL(v)}(T^*Q)$, then $\Omega(T(L)_v X, Y) = 0$ because both $TL_v(X)$ and Y are vertical. On the other side, $R(L)_v(Y) = 0$ and so $T(L)_v Y = 0$, too. The relation (5.6) is then true, in this case.

ii) Let now assume that $Y \in FL*_v T(TQ)$. There is a decomposition of X as a sum $X = X_1 + X_2$ with $X_1 \in FL*_v T(TQ)$, $X_2 \in V_{FL(v)}(T^*Q)$. Let $Y' \in T_v(TQ)$ be such that $FL*_v Y' = Y$ and similarly a $X_1' \in T_v(TQ)$ is chosen such that $FL*_v X_1' = X_1$. Then $\Omega(T(L)_v X_1, Y)$ can be rewritten

$$\Omega(T(L)_v X_1, Y) = \Omega(T(L)_v \circ FL*_v X'_1, FL*_v Y'),$$

and taking into account (5.4) we will obtain

$$\Omega(T(L)_v X_1, Y) = \Omega(FL*_v \circ S_v X'_1, FL*_v Y') = \omega_L(S_v X'_1, Y'),$$

and when using (2.3a) it becomes

$$\Omega(T(L)_v X_1, Y) = \omega_L(S_v Y', X'_1) = \Omega(FL*_v S_v Y', X_1).$$

Finally, the intertwining relation (5.5) leads to

$$\Omega(T(L)_v X_1, Y) = \Omega(T(L)_v Y, X_1).$$

As far as the other component X_2 is concerned, $\Omega(T(L)_v X_2, Y) = 0$, because $R(L)_v X_2 = 0$, and $\Omega(X_2, T(L)_v Y)$ vanishes too, because both fields are vertical. ∎

We will now be interested in the relations between the constraint functions in the Lagrangian formulation and those of the Hamiltonian one. This relations are going to be studied by means of the map $R(L)$ introduced in the beginning of this Section. Following the notation of [6], the primary constraint submanifold in the Hamiltonian formulation will be denoted $M_1 = FL(TQ)$ and we will hereafter assume that the Lagrangian system is almost regular, i.e. FL is a submersion onto its image and for every $v \in TQ$ the fibres $FL^{-1}\{FL(v)\}$ are assumed to be connected submanifolds of TQ. We also recall [8, 9, 24] that the constraint functions ϕ defining M_1 are those with associated vector fields taking their values in $TM_1^{\perp}$, while first class (at the M_1 level) constraint functions correspond to vector fields with values in $TM_1 \cap (TM_1)^{\perp}$. Hence, we will next start with the study of the image of $[FL*_v T(TQ)]^{\perp}$ under $R(L)_v$.

THEOREM 2. *The image under* $R(L)_v$ *of* $[FL*_v T(TQ)]^{\perp}$ *coincides with* $V_v(\mathrm{Ker}\,\omega_L)$.

Proof. Let X be an element of $[FL*_v T(TQ)]^{\perp}$. Then, $\forall Y \in T_v(TQ)$,

$$\omega_L(R(L)_v X, Y) = \Omega(FL*_v \circ R(L)_v X, FL*_v Y) = \Omega(T(L)_v X, FL*_v Y),$$

and we can make use of the former proposition for obtaining that

$$\omega_L(R(L)_v X, Y) = -\,\Omega(X, T(L)_v \circ FL*_v Y) = -\,\Omega(X, FL*_v \circ S_v Y) = 0,$$

where we have also taken into account the intertwining property (5.5) between $T(L)_v$ and S_v. Thus, $R(L)_v[FL*_v T(TQ)]^{\perp}$ is contained in $V(\mathrm{Ker}\,\omega_L)$.

Furthermore the restriction of $R(L)_v$ to $[FL*_v T(TQ)]^{\perp}$ is injective because of (3.2). Then $R(L)_v[FL*_v T(TQ)]^{\perp} = V_v(\mathrm{Ker}\,\omega_L)$. ∎

As regards the theory of constraint functions, this theorem says that if ϕ is a constraint function for the M_1 submanifold, then $R(L)X_\phi$ is in $V(\text{Ker } \omega_L)$ and that this last set is generated in this way. In particular, if we recall (3.3), that means that the vector fields $R(L)X_\phi$ annihilate the FL-projectable functions [18].

PROPOSITION 8. *The map $T(L)_v$ has the following properties*

i) $X \in \text{Ker } T(L)_v$ if and only if $R(L)_v X \in V(\text{Ker } \omega_L)$.

ii) The kernel of $T(L)_v$ coincides with $[ImT(L)_v]^\perp$.

iii) For every $v \in T(TQ)$, the vector subspace $ImT(L)_v$ is isotropic.

Proof. i) Let us supose that $X \in \text{Ker } T(L)_v$. Then, $R(L)_v X$ is vertical and furthermore, for any $Y \in T_v(TQ)$,

$$\omega_L(R(L)_v X, Y) = \Omega(FL*_v \circ R(L)_v X, FL*_v Y) = \Omega(T(L)_v X, FL*_v Y) = 0$$

Conversely, let X be such that $R(L)_v X \in V(\text{Ker } \omega_L)$. Then, for an arbitrary $Y \in T_v(TQ)$, we have $\omega_L(R(L)_v X, Y) = 0$ and therefore $\Omega(FL*_v \circ R(L)_v X, FL*_v Y) = 0$. From this relation we see that $T(L)_v X \in [FL*_v T(TQ)]^\perp$ and as $T(L)_v X$ is vertical, the relation (3.2) shows that $X \in \text{Ker } T(L)_v$.

ii) It is a consequence of (5.6)

iii) We first remark that $[T(L)_v]^2 = 0$ because

$$[T(L)_v]^2 = FL*_v \circ R(L)_v \circ FL*_v \circ R(L)_v = FL*_v \circ S \circ R(L)_v = 0.$$

Hence $ImT(L)_v$ is contained in Ker $T(L)_v$. Now, ii) implies that $ImT(L)_v$ is an isotropic subspace of $T_{FL(v)}(T^*Q)$. ∎

PROPOSITION 9. *The subspace* Ker $T(L)_v$ *is a direct sum*

$$\text{Ker } T(L)_v = V_{FL(v)}(T^*Q) \oplus [FL*_v T(TQ)]^\perp$$

Proof. It is based on the preceding Proposition, because $R(L)_v\{V_{FL(v)}(T^*Q)\} = 0$ and it has been shown in Theorem 2 that $R(L)_v\{[FL*_v T(TQ)]^\perp\} = V_v(\text{Ker } \omega_L)$. Hence, it follows the relation $V_{FL(v)}(T^*Q) \oplus [FL*_v T(TQ)]^\perp \subset \text{Ker } T(L)_v$.

Furthermore, if X is an element of Ker $T(L)_v$, let Y be the element of $V_v(\text{Ker } \omega_L)$ defined by $R(L)_v X = Y$. Then, Theorem 4 shows the existence of an element $X_2 \in [FL*_v T(TQ)]^\perp$ such that $Y = R(L)_v X_2$. The difference $X_1 = X - X_2$ is vertical and then $X \in V_{FL(v)}(T^*Q) \oplus [FL*_v T(TQ)]^\perp$. ∎

An important property of the first class (at the M_1-level) constraint functions is given in the following theorem:

THEOREM 3. *Let ϕ be a constraint function for M_1 and X_ϕ denote the corresponding vector field $X_\phi = \hat{\Omega}^{-1}$ (dϕ). Then, ϕ is first class (at the M_1 level) if and only if there exists a $Z_\phi \in \operatorname{Ker} \omega_L$ such that $S(Z_\phi) = R(L)X_\phi$.*

Proof. If ϕ is first class, there will be a FL-projectable vector field $Z_\phi \in \mathscr{X}(TQ)$ such that $FL{*}Z_\phi = X_\phi$ (see e.g. [6]). Then,

$$R(L)X_\phi = R(L)FL{*}Z_\phi = S(Z_\phi).$$

The vector field Z_ϕ built in this way lies in Ker ω_L because, in a pointwise sense

$$\omega_L(Z_\phi, U) = \Omega(FL{*}Z_\phi, FL{*}U) = \Omega(X_\phi, FL{*}U) = 0$$

the last identity being a consequence of X_ϕ being a vector field with values in $[FL{*}T(TQ)]^\perp$.

Conversely, if we assume the existence of a vector field $Z_\phi \in \operatorname{Ker} \omega_L$ such that $S(Z_\phi) = R(L)X_\phi$. The difference $FL{*}_v Z_\phi(v) - X_\phi(FL(v))$ is vertical and lies in $[FLT(TQ)]^\perp$ because for $U = FL{*}_v W$, in a pointwise sense, we can write

$$\Omega(FL{*}Z_\phi - X_\phi, U) = \omega_L(Z_\phi, W) - \Omega(X_\phi, U)$$

and the first term on the right hand side vanishes because $Z_\phi \in \operatorname{Ker} \omega_L$ while the second vanishes too because ϕ is a constraint function. Finally, from (3.2) we see that $X_\phi = FL{*}Z_\phi$ and then X_ϕ takes its values in $TM_1 \cap (TM_1)^\perp$ so that ϕ is first class. ∎

As a corollary of this theorem we can obtain in a different way a result given in [21]:

COROLLARY 3. *The primary constraint submanifold M_1 is coisotropic if and only if L is a type II Lagrangian.*

Proof. If M_1 is coisotropic all the constraint functions for M_1 are first class. But the vector fields $R(L)X_\phi$ generate $V(\operatorname{Ker} \omega_L)$ and for such vector fields we can choose $Z \in \operatorname{Ker} \omega_L$ such that $S(Z) = R_v X_\phi$. That means that $S' \equiv S|_{\operatorname{Ker} \omega_L}$ is onto $V(\operatorname{Ker} \omega_L)$. Conversely, if L is of type II, the map S' is onto and the preceding theorem shows that every constraint function will be a first class constraint function. ∎

All these last properties can also be seen as a consequence of the following property:

PROPOSITION 10. *Let ϕ and ψ be two constraint functions for M_1 and Y_ϕ, $Y_\psi \in \chi(TQ)$ such that $S(Y_\phi) = R(L)X_\phi$ and $S(Y_\psi) = R(L)X_\psi$. Then,*

$$\omega_L(Y_\phi, Y_\psi) = -\Omega(X_\phi, X_\psi) \circ FL = -\{\phi, \psi\} \circ FL \tag{5.7}$$

where $\{\phi, \psi\}$ is the Poisson bracket of the constraint functions.

Proof. The differences $FL_*Y_\phi - X_\phi$ and $FL_*Y_\psi - X_\psi$ (in a pointwise sense) are vertical and therefore

$$\Omega(FL_*Y_\phi - X_\phi, FL_*Y_\psi - X_\psi) = 0$$

Consequently, if we recall that X_ϕ and X_ψ take values in $TM_1^\perp$ we obtain the relation (5.7). ∎

The results of this section allow to establish a correspondence between primary Hamiltonian and Lagrangian constraints, different from the usual FL-pull-back, using the map R_L as follows:

i) Given a primary Hamiltonian constraint ϕ, and the related vector field $X_\phi = \hat{\Omega}^{-1}(d\phi)$, we associate to ϕ the primary Lagrangian constraint function

$$\chi_\phi = (\hat{\omega}_L(\Gamma_0) - dE_L)(Y_\phi), \tag{5.8}$$

where Y_ϕ is any vector field in TQ such that $S(Y_\phi) = R_L(X_\phi) \in V(\text{Ker}\,\omega_L)$ and Γ_0 is any S.O.D.E..

ii) First class primary Hamiltonian constraint are associated in that way with primary dynamical Lagrangian constraints, while 2^{nd} class primary Hamiltonian constraints are associated with primary S.O.D.E. conditions. In fact, the results of Theorem 5 say that if ϕ is a first class function, the Y_ϕ in (5.8) can be chosen in Ker ω_L and then such a constraint is a dynamical constraint in TQ. On the contrary, if ϕ is of the second class, there will exist a primary constraint ψ such that $\{\psi, \phi\} \circ FL \neq 0$, and consequently (5.7) shows that there is no vector field $Y_\phi \in \text{Ker}\,\omega_L$ such that $S(Y_\phi) = X_\phi$ and the corresponding constraint (5.8) will be not a dynamical constraint but a S.O.D.E. condition.

We can also ask whether the Lagrangian constraints so obtained are FL-projectable. Actually, if ϕ is a first class primary function, then (5.8) reduces to $dE_L(Y_\phi)$. But we recall that $H \circ FL = E_L$, and we can therefore rewrite the expression (5.8) as

$$\chi_\phi = dH \circ FL_*(Y_\phi) \circ FL = FL^*(X_\phi(H)),$$

which shows that the constraint χ_ϕ given by (5.8) is projectable; moreover, it is the pull-back of the corresponding secondary Hamiltonian constraint.

iii) In some cases the functions χ_ϕ may become identities, for instance, when there are no secondary Hamiltonian constraints the dynamical Lagrangian constraints reduce to identities, but S.O.D.E. conditions may still remain.

6. APPLICATIONS

In order to make the theory ready for explicit calculations we give the coordinate expressions for the different results of the paper.

Take coordinates q^i on the configuration space Q and correspondingly q^i, v^i on TQ (with range and sum conventions in force). The map $\hat{\omega}_L$ when referred to the corresponding local basis of $\mathscr{X}(TQ)$ and $\Lambda^1(TQ)$ is

$$\hat{\omega}_L = \begin{bmatrix} A & -W \\ W & 0 \end{bmatrix}$$

where W and A are, respectively,

$$A_{ij} = \partial^2 L/\partial q^i \partial v^j - \partial^2 L/\partial v^i \partial q^j \quad \text{and} \quad W_{ij} = \partial^2 L/\partial v^i \partial v^j. \tag{6.1}$$

The elements $X = \xi^i \partial/\partial v^i$ of $V(\mathrm{Ker}\ \omega_L)$ are determined by the condition

$$W\xi = 0, \tag{6.2}$$

and consequently the vector fields in TQ with values in M will be of the form $\xi^i \partial/\partial q^i + \eta^i \partial/\partial v^i$ with ξ satisfying (6.2) and η being arbitrary.

We recall that the set of points of TQ in which there exists a solution of the dynamical equation (2.1) that is the restriction of a S.O.D.E. was determined by the conditions (4.3). The one-form $i(\Gamma_0)\omega_L - \mathrm{d}E_L$ being a semibasic form, these conditions may also be rewritten

$$< \xi, \alpha > = 0 \quad \forall \xi \text{ such that } W\xi = 0, \tag{6.3}$$

where

$$\alpha_i = \partial L/\partial q^i - (\partial^2 L/\partial v^i \partial q^j) v^j, \tag{6.4}$$

no matter of the choice for the S.O.D.E. Γ_0.

As remarked before, the case in which the vector field X in (4.3) can be chosen in Ker ω_L is that of «dynamical constraints» while the remaining equations will be S.O.D.E. conditions. The difference between both cases is also clear in coordinates. Given a vector field $X = \xi^i \partial/\partial q^i + \eta^i \partial/\partial v^i$ such that $S(X) \subset M$ (i.e. such that $W\xi = 0$), there will exist a vector field $Y = \xi^i \partial/\partial q^i + \zeta^i \partial/\partial v^i$ in Ker ω_L if and only if

$$< \xi', A\xi > = 0 \qquad \forall \xi' \text{ such that } W\xi' = 0, \tag{6.5}$$

as a direct consequence of the Theorem 1, so that the set of the dynamical constraint functions» is the subset of (6.3) corresponding to those ξ such that condition (6.5) holds, the remaining constraint functions being S.O.D.E. conditions.

As far as the connection between the constraint functions arising in both formulations is concerned we want to point out that if ϕ is a function in T^*Q and f is a function in TQ, the coordinate expression for $R(L)X_\phi(f)$ is

$$(6.6)\qquad R(L)X_\phi(f)(v) = (\partial\phi/\partial p_i)|_{FL(v)}(\partial f/\partial v^i)|_v.$$

Next some simple examples which display the different aspects of the theory are analysed. As a first example we will study the one proposed by Christ and Lee [25], in its reduced form:

$$(6.7)\qquad L = 1/2\ (\dot{r}^2 + r^2(\dot{\theta} - z)^2] - V(r) \qquad (r \neq 0),$$

which has also been studied by Costa and Girotti [26]. In this case the Legendre transformation is given by

$$(6.8)\qquad p_r = \dot{r}, \quad p_\theta = r^2(\dot{\theta} - z) \quad \text{and} \quad p_z = 0$$

and therefore there is one primary constraint function $\phi_1 = p_z$ (obviously first class). Then, the corresponding constraint function in the Lagrangian formalism is $\alpha_3 = 0$, namely, $\chi = r^2(\dot{\theta} - z)$. This constraint is dynamical: in fact, Ker W is one-dimensional and no S.O.D.E. conditions will arise. Furthermore, the function χ is FL-projectable with $\chi = FL^*(\phi_2)$ with $\phi_2 = p_\theta$. Thus, both ϕ_1 and ϕ_2 are related with χ, ϕ_1 by the procedure proposed in this paper while ϕ_2 by FL-pull-back. It is notweworthy that ϕ_2 is the secondary constraint obtained by imposing the stability of the constraint ϕ_1 under time evolution.

Another example, proposed by Nesterenko and Chervyakov [19] is

$$(6.9)\qquad L = 1/2\ (v_1{}^2) - v_2 x_3.$$

In this example Ker W is two-dimensional and the inner product $< \xi', A\xi >$ is different from zero for any two linearly independent vectors of Ker W. Therefore, there will be no dynamical constraints in the Lagrangian formulation but just the S.O.D.E. conditions

$$(6.10)\qquad \alpha_2 = v_3 = 0 \quad \text{and} \quad \alpha_3 = -v_2 = 0$$

In the Hamiltonian formulation there are two primary constraint functions which are of the second class. The corresponding Lagrangian constraints are those given by (6.10) and they are not projectable. There are no secondary constraints in the Hamiltonian formulation.

As a final example we will study a Poincaré-like model in which a new variable is added. The configuration space is the open manifold obtained from

$\mathbb{R}^5 = \{(x^1, x^2, x^3, x^0, A)\}$ by removing the hyperplane $A = 0$. The Lagrangian is assumed to be the following singular one

(6.11) $$L = 1/2 \cdot \{v_\mu v^\mu /A + m^2 A\}$$

It is easy to check that Ker W is one-dimensional and Ker ω_L is generated by $\partial/\partial V$ and $\partial/\partial A + (v^\mu/A)\,\partial/\partial v^\mu$, with V denoting the velocity of the coordinate A. In this case there is no S.O.D.E. condition (note that it satisfies the extremal condition for dimensions of Ker ω_L and V(Ker ω_L)) because Ker W being one-dimensional the condition (6.5) always holds. The only primary constraint in this Lagrangian formulation will be $\alpha_5 = 0$, namely,

$$\chi = \partial L/\partial A = -\{v_\mu v^\mu /A^2\} + m^2.$$

The general solution for the dynamical equation (2.1) is

$$\Gamma = v^\mu \partial/\partial q^\mu + \lambda\{\partial/\partial A + (v^\mu/A)\,\partial/\partial v^\mu\}$$

which explicitly shows that there will exist no S.O.D.E. condition: it is always possible to choose λ equal to V.

The Legendre transformation is defined by

$$p^\mu = v^\mu/A, \qquad \Pi = 0$$

where Π is the momentum conjugate to A. Here there is only the primary constraint $\phi_1 = \Pi$. Furthermore, there will be a secondary constraint $\phi_2 = 1/2(p^\mu p_\mu - m^2)$, obtained either by a geometrical procedure following the Gotay algorithm or using the classical Dirac theory. The point we want to stress here is that both constraint functions are of the first class and "$R(L)\phi_1 = \chi_1 = \alpha_5$" which is the Lagrangian constraint we found directly in the Lagrangian formulation. Moreover, the constraint χ_1 is FL-projectable, with χ_1 being $\chi_1 = FL^*\phi_2$, as expected according to the general theory, because ϕ_2 is the only secondary constraint.

CONCLUSIONS

The introduction of a new operator $R(L) : \mathfrak{X}(T^*Q) \to \mathfrak{X}(TQ)$ is used to find a relation between the Hamiltonian constraints and all the Lagrangian ones, at the first level of the constrain algorithm. It is shown why the former theory developed by Gotay and Nester [5] can not deal with the S.O.D.E. conditions (non FL-projectable constraints). The parallelism between first class Hamiltonian constraints with dynamical Lagrangian ones and second class Hamiltonian constraints with S.O.D.E. conditions is studied with the new optics. However, the theory here developed is not complete, and the study at higher levels of the constraint algorithm makes necessary the introduction of a new operator, the K operator [27]. Using both operators, K and $R(L)$, the main results of the paper can be generalized to all the levels of the constraint algorithm.

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